J. Fluid Mech. (1977), *vol.* 83, *part* 1, *pp.* 141–151
Printed in Great Britain

Long surface waves incident on a submerged horizontal plate

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(Received 28 October 1976 and in revised form 16 March 1977)

A train of surface gravity waves of wavelength λ in a channel of depth H is incident on a horizontal plate of length l that is submerged to a depth c. Under the assumption that both λ and l are large compared with H, the method of matched asymptotic expansions is used to show that, to first order, the reflexion coefficient R and the transmission coefficient T are given by

$$R = \chi\left\{\frac{\sigma l}{(gH)^{\frac{1}{2}}}\sin\frac{\sigma l}{(gc)^{\frac{1}{2}}} - 2\left(\frac{c}{H}\right)^{\frac{1}{2}}\left(1-\cos\frac{\sigma l}{(gc)^{\frac{1}{2}}}\right)\right\}$$

and

$$T = \chi\left\{2i\left[\sin\frac{\sigma l}{(gc)^{\frac{1}{2}}} + \frac{\sigma l}{b}\left(\frac{c}{g}\right)^{\frac{1}{2}}\right]\right\},$$

where

$$\chi = 1\Big/\left\{2\left(\frac{c}{H}\right)^{\frac{1}{2}}\left(1-\cos\frac{\sigma l}{(gc)^{\frac{1}{2}}}\right) + \frac{\sigma l}{b}\left(\frac{H}{g}\right)^{\frac{1}{2}}\left(1+\frac{c}{H}\right)\sin\frac{\sigma l}{(gc)^{\frac{1}{2}}} + 2i\left(\sin\frac{\sigma l}{(gc)^{\frac{1}{2}}} + \frac{\sigma l}{b}\left(\frac{c}{g}\right)^{\frac{1}{2}}\cos\frac{\sigma l}{(gc)^{\frac{1}{2}}}\right)\right\},$$

σ is the angular frequency and g the acceleration due to gravity.

1. Introduction

This investigation is motivated by the desire to build breakwaters that are buoyant and are of a less permanent nature. A start on the problem was made by Heins (1950) and later Greene & Heins (1953), who considered semi-infinite barriers. A related problem is the floating finite dock. This was considered by Holford (1964) for the short-wave limit using an approximate kernel in an integral-equation approach. This problem was solved again, this time through matched asymptotic expansions, by Leppington (1972) and Hermans (1972). Stoker (1957, p. 430) and Wells (1953) discussed the problem in terms of shallow-water theory.

Burke (1964) considered a submerged finite plate in water of infinite depth using a Wiener–Hopf technique. This result was generalized to the case of finite depth in Siew (1976).

In this investigation we consider the effect on a train of waves of wavelength λ of a horizontal plate of length l that is submerged to a depth c in a channel of depth H. The method of matched asymptotic expansions is used to determine the reflexion coefficient R and the transmission coefficient T in the limit $\epsilon = \kappa H \to 0$ with κl fixed. Here $\kappa = \sigma/(gH)^{\frac{1}{2}}$ is the wavenumber according to shallow-water theory, σ being the angular frequency and g the acceleration due to gravity, and the limit corresponds to both λ and l being large compared with H. In this approach there are four 'outer' regions: one far upstream of the plate, one far downstream and two above and below the plate respectively that are far from both its edges. The solutions in these regions are determined in §2. The vicinity of either edge of the plate is an 'inner' region and the solutions here are determined in §3. The matching of the inner and outer solutions is accomplished in §§4 and 5. Section 6 outlines a scheme for obtaining higher approximations and the results are discussed in §7. Section 8 gives an extension to the case of oblique incidence.

2. The outer solutions

In terms of the velocity potential Φ, the governing equations are

$$\left.\begin{aligned} \Phi_{xx}+\Phi_{yy} &= 0, \\ \Phi_y-(\sigma^2/g)\,\Phi &= 0 \quad \text{on} \quad y = H, \\ \Phi_y &= 0 \quad \text{on} \quad \begin{cases} y=0, \\ y=b, & |x| < a, \end{cases} \end{aligned}\right\} \tag{2.1}$$

where the origin of the rectangular axes Oxy is taken at the bottom of the channel, with y increasing to H at the free surface. The plate occupies $|x| < a, y = b$ $(0 < b < H)$, and a time-dependent factor $e^{-i\sigma t}$ is assumed throughout.

It is well known that when the wavelength λ is large compared with the depth H the phase speed is approximately $(gH)^{\frac{1}{2}}$ and a wavelength scale is $(gH)^{\frac{1}{2}}/\sigma$. In $x \gg a$ we therefore define the non-dimensional co-ordinates

$$(X, Y) = \left(\frac{\sigma}{(gH)^{\frac{1}{2}}}(x-a), \frac{y}{H}\right),$$

with a small parameter $\epsilon = \sigma(H/g)^{\frac{1}{2}} = O(H/\lambda)$. Equations (2.1) then become

$$\left.\begin{aligned} \Phi_{YY}+\epsilon^2\Phi_{XX} &= 0, \\ \Phi_Y-\epsilon^2\Phi &= 0 \quad \text{on} \quad Y = 1, \\ \Phi_Y &= 0 \quad \text{on} \quad Y = 0. \end{aligned}\right\} \tag{2.2}$$

In $x \ll -a$, we can similarly define

$$(X_1, Y_1) = \left(\frac{\sigma}{(gH)^{\frac{1}{2}}}(x+a), \frac{y}{H}\right)$$

and show that the same differential system (2.2) is obtained. This is also true of the region above the plate (but away from the edges), where we shall define

$$(\xi, \eta) = (\sigma x/(gH)^{\frac{1}{2}}, y/H).$$

We shall denote by Φ^L the solution appropriate for the region $x \ll -a$, by Φ^R the solution for $x \gg a$ and by Φ^U and Φ^D the solutions for the regions above and below the plate.

The leading-order solution of (2.2) may be obtained easily and from it the form of the inner expansion will be evident. In the spirit of the matched asymptotic method the leading term(s) of the inner solution will in turn suggest the order of the next term in the outer solution, and so on. We shall assume here, however, that Φ has an expansion of the form

$$\Phi = \Phi_0+\epsilon\Phi_1+\epsilon^2\Phi_2+\dots, \tag{2.3}$$

which may be easily verified from the form of the inner expansions obtained in the next two sections.

In $x \gg a$, then, (2.2) leads to the systems

$$\left.\begin{aligned} \Phi^R_{mYY} &= 0, \\ \Phi^R_{mY} &= 0 \quad \text{on} \quad Y = 0, 1, \end{aligned}\right\} \quad m = 0, 1, \tag{2.4}$$

and

$$\left.\begin{aligned} \Phi^R_{nYY} &= -\Phi^R_{n-2XX}, \\ \Phi^R_{nY} &= \begin{cases} \Phi^R_{n-2} & \text{on} \quad Y = 1, \\ 0 & \text{on} \quad Y = 0, \end{cases} \end{aligned}\right\} \quad n = 2, 3, 4, \dots . \tag{2.5}$$

The representations for Φ^R and hence Φ^L and Φ^U are immediate and may be written as

$$\begin{aligned} \Phi^L = {} & \exp(iX_1) + R_0 \exp(-iX_1) + \epsilon R_1 \exp(-iX_1) \\ & + \epsilon^2 \left[\left(\frac{iX_1}{6} + \frac{Y_1^2}{2} \right) \exp(iX_1) + \left\{ R_2 - R_0 \left(\frac{iX_1}{6} - \frac{Y_1^2}{2} \right) \right\} \exp(-iX_1) \right] + \dots, \end{aligned} \tag{2.6}$$

$$\Phi^R = T_0 \exp(iX) + \epsilon T_1 \exp(iX) + \epsilon^2 \left[T_2 + T_0 \left(\frac{iX}{6} + \frac{Y^2}{2} \right) \right] \exp(iX) + \dots \tag{2.7}$$

and

$$\begin{aligned} \Phi^U = {} & U_0 \exp[i(H/c)^{\frac{1}{2}} \xi] + V_0 \exp[-i(H/c)^{\frac{1}{2}} \xi] + \epsilon \{ U_1 \exp[i(H/c)^{\frac{1}{2}} \xi] + V_1 \exp[-i(H/c)^{\frac{1}{2}} \xi] \} \\ & + \epsilon^2 \left[\left\{ U_2 + U_0 \left[\frac{i}{6} \left(\frac{c}{H} \right)^{\frac{1}{2}} \xi + \frac{\eta^2 H}{2c} - \frac{\eta b}{c} \right] \right\} \exp[i(H/c)^{\frac{1}{2}} \xi] \right. \\ & \left. + \left\{ V_2 + V_0 \left[\frac{\eta^2 H}{2c} - \frac{i}{6} \left(\frac{c}{H} \right)^{\frac{1}{2}} \xi - \frac{\eta b}{c} \right] \right\} \exp[-i(H/c)^{\frac{1}{2}} \xi] \right] + \dots . \end{aligned} \tag{2.8}$$

In (2.6)–(2.8), we have assumed a wave incident from $X_1 = -\infty$ with amplitude $1 + O(\epsilon^2)$, that R_n, T_n, U_n and V_n ($n = 0, 1, 2, \dots$) are constants and that $c = H - b$. (It is noted that for Φ^U we could use $\sigma/(gc)^{\frac{1}{2}}$ as the scale for the x co-ordinate; however it is clear that the same solution would be obtained and so the one horizontal length scale is used in each of the outer representations.)

We note here that the vertical velocity component comes into the solution only through terms of order ϵ^2 and higher. Further, the dispersion relation is

$$\kappa H \tanh \kappa H = \sigma^2 H / g = \epsilon^2,$$

which gives $\kappa H = \epsilon + \frac{1}{6}\epsilon^3 + \dots$. Consider a wave term given by

$$\phi_i = \exp[i\kappa(x + a)] \cosh \kappa y;$$

writing κH in terms of ϵ and expanding leads to

$$\phi_i = \exp(iX_1) \{ 1 + \epsilon^2 (\tfrac{1}{6} i X_1 + \tfrac{1}{2} Y_1^2) + \dots \}.$$

Thus the occurrence of polynomials (in X_1 and Y_1) in the coefficients of the exponential terms in (2.6)–(2.8) is to be expected and is a consequence of expanding in powers of ϵ.

In $|x| < a$, $0 < y < b$ (under the plate) equations (2.1) become

$$\left.\begin{aligned} \Phi^D_{xx} + \Phi^D_{yy} &= 0, \\ \Phi^D_y &= 0 \quad \text{on} \quad y = 0, b. \end{aligned}\right\} \tag{2.9}$$

These can be solved by separation of variables, and hence we have

$$\Phi^D = P_0\xi_1 + Q_0 + \sum_1^\infty \left\{P_n \exp\left(\frac{n\pi a}{b}\xi_1\right) + Q_n \exp\left(-\frac{n\pi a}{b}\xi_1\right)\right\} \cos\frac{n\pi y}{b}, \tag{2.10}$$

where $\xi_1 = x/a$ and P_n and Q_n $(n = 0, 1, 2, \ldots)$ are constants.

3. The inner solutions

Close to the edge (a, b) the relevant length scale is H and on putting $\tilde{x} = (x-a)/H$ and $\tilde{y} = y/H$ $(X = \epsilon\tilde{x})$ we obtain from (2.1)

$$\left.\begin{array}{c} \phi_{\tilde{x}\tilde{x}} + \phi_{\tilde{y}\tilde{y}} = 0, \\ \phi_{\tilde{y}} - \epsilon^2\phi = 0 \quad \text{on} \quad \tilde{y} = 1, \\ \phi_{\tilde{y}} = 0 \quad \text{on} \quad \begin{cases} \tilde{y} = 0, \\ \tilde{y} = b/H, \quad -2a/H < \tilde{x} < 0. \end{cases} \end{array}\right\} \tag{3.1}$$

From (2.7), on putting $X = \epsilon\tilde{x}$ and expanding in the limit $\epsilon \to 0$ we find that

$$\Phi^R \sim T_0 + \epsilon[i\tilde{x}T_0 + T_1] + O(\epsilon^2),$$

which suggests an inner development of the form

$$\phi \sim \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots . \tag{3.2}$$

In (3.1), if we consider plate widths of the order of the wavelength then

$$2a/H = O(\lambda/H) = O(1/\epsilon),$$

and since the matching with the outer limit should be smooth in the limit $\epsilon \to 0$ it is natural to apply the boundary condition on the plate for all negative $\tilde{x}$. Substituting (3.2) into (3.1) then yields

$$\left.\begin{array}{c} \phi_{m\tilde{x}\tilde{x}} + \phi_{m\tilde{y}\tilde{y}} = 0, \\ \phi_{m\tilde{y}} = 0 \quad \text{on} \quad \begin{cases} \tilde{y} = 0, 1, \\ \tilde{y} = b/H, \quad \tilde{x} < 0, \end{cases} \end{array}\right\} \quad m = 0, 1, \tag{3.3}$$

and

$$\left.\begin{array}{c} \phi_{n\tilde{x}\tilde{x}} + \phi_{n\tilde{y}\tilde{y}} = 0, \\ \phi_{n\tilde{y}} = \phi_{n-2} \quad \text{on} \quad \tilde{y} = 1, \\ \phi_{n\tilde{y}} = 0 \quad \text{on} \quad \begin{cases} \tilde{y} = 0, \\ \tilde{y} = b/H, \quad \tilde{x} < 0, \end{cases} \end{array}\right\} \quad n = 2, 3, \ldots . \tag{3.4}$$

The conditions at the two infinities will be replaced by the matching requirements as discussed in §5.

Close to the edge $(-a, b)$ we let $\tilde{x}_1 = (x+a)/H$ and $\tilde{y}_1 = y/H$ $(X_1 = \epsilon\tilde{x}_1)$ and hence the solution can be obtained from (3.3) and (3.4) by replacing $\tilde{x}$ with $-\tilde{x}_1$.

Now, the solution to (3.3) may be obtained through a Schwarz–Christoffel mapping as depicted in figure 1, where the complex $\tilde{z}$ plane is mapped onto the upper half of the ζ plane. This is accomplished by the mapping defined by

$$d\tilde{z}/d\zeta = k(\zeta + d)^{-1}\zeta(\zeta - 1)^{-1}, \tag{3.5}$$

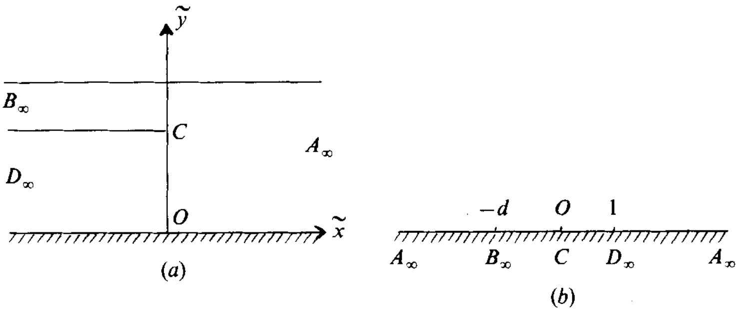

FIGURE 1. (a) $\tilde{z}$ plane ($\tilde{z} = \tilde{x}+i\tilde{y}$), (b) ζ plane.

whence
$$\tilde{z} = \frac{b}{\pi H}\{\ln(\zeta-1)+d\ln(\zeta+d)\}-\frac{c}{\pi H}\ln\frac{c}{b}, \quad d=\frac{c}{b}, \tag{3.6}$$

where the constants k and d have been determined by the usual method.

In terms of the complex potential $W(\zeta) = \phi+i\psi$, which is such that Im W is constant along the boundaries of the region, we have

$$W_0(\zeta) = \frac{Q_a}{\pi}\ln(\zeta+d)+\frac{Q_b}{\pi}\ln(\zeta-1)+F \tag{3.7}$$

and
$$W_1(\zeta) = \frac{Q_a'}{\pi}\ln(\zeta+d)+\frac{Q_b'}{\pi}\ln(\zeta-1)+F', \tag{3.8}$$

where Q_a, Q_b, F, Q_a', Q_b' and F' are constants.

Im W constant along $A_\infty B_\infty$ requires Q_a and Q_b (and Q_a' and Q_b') to be real, while if we put $\psi = 0$ along the bottom of the channel F (and F') also should be real.

4. The outer expansion of the inner solution

In anticipation of the matching of the solutions in the last two sections we shall now proceed to obtain the form of the solutions ϕ_0 and ϕ_1 in the limit $\tilde{x}\to\infty$ with X fixed (where $\tilde{x} = X/\epsilon$). From figure 1 we see that $\tilde{x}\to\infty$ corresponds to $|\zeta|\to\infty$ in the ζ plane and from (3.6) we have

$$\tilde{z} = \frac{1}{\pi}\ln\zeta-\frac{c}{\pi H}\ln\frac{c}{b}-\frac{b-c}{\pi b}\frac{1}{\zeta}+O\left(\frac{1}{\zeta^2}\right) \quad \text{for} \quad |\zeta|\gg 1,$$

whence it can be shown that

$$\zeta \sim \exp\left[\pi\tilde{z}+\frac{c}{H}\ln\left(\frac{c}{b}\right)\right]+\frac{b-c}{b}+O(\exp(-\pi\tilde{z})) \quad \text{for} \quad |\tilde{z}|\gg 1.$$

Equation (3.7) then gives

$$W_0 \sim \frac{Q_a+Q_b}{\pi}\left(\pi\tilde{z}+\frac{c}{H}\ln\frac{c}{b}\right)+F+O(\exp(-\pi\tilde{z})) \quad \text{for} \quad \tilde{x}\gg 1, \quad 0<\tilde{y}<1. \tag{4.1}$$

The real part of (4.1) yields

$$\phi_0 = \frac{Q_a+Q_b}{\pi}\left(\pi\tilde{x}+\frac{c}{H}\ln\frac{c}{b}\right)+F+O(\exp(-\pi\tilde{x})) \quad \text{for} \quad \tilde{x}\gg 1, \tag{4.2}$$

which, in terms of the outer variable X, is

$$\phi_0 = \frac{Q_a+Q_b}{\pi}\left(\frac{\pi X}{\epsilon}+\frac{c}{H}\ln\frac{c}{b}\right)+F+O(\exp(-\pi X/\epsilon)).$$

It is obvious that for matching with (2.7) to be possible $(Q_a+Q_b)X/\epsilon$ must be zero. On the other hand, when $\tilde{x} \ll -1$ and $b/H < \tilde{y} < 1$ we are over the plate and ζ is near B_∞ in figure 1(b). Putting $|\zeta+d| \ll 1$ in (3.6) and expanding as before leads to

$$W_0(\tilde{z}) = \frac{Q_aH}{c}\left\{\left(\tilde{z}-\frac{ib}{H}\right)+\frac{c}{\pi H}\ln\frac{c}{b}-\frac{b}{\pi H}\ln\frac{H}{b}\right\}$$
$$+\frac{Q_b}{\pi}\left(i\pi+\ln\frac{H}{b}\right)+F+O(\exp(\pi H\tilde{z}/c)) \quad \text{for} \quad \tilde{x} \ll -1, \quad \frac{b}{H} < \tilde{y} < 1,$$

the real part of which gives

$$\phi_0 = \frac{Q_aH}{c}\left(\tilde{x}+\frac{c}{\pi H}\ln\frac{c}{b}-\frac{b}{\pi H}\ln\frac{H}{b}\right)+\frac{Q_b}{\pi}\ln\frac{H}{b}+F$$
$$+O(\exp(\pi H\tilde{x}/c)) \quad \text{for} \quad \tilde{x} \ll -1, \quad \frac{b}{H} < \tilde{y} < 1. \quad (4.3)$$

This should match with Φ^U near $\xi = \sigma a/(gH)^{\frac{1}{2}}$ $(x = a)$. Now we have

$$\xi = \epsilon\tilde{x}+\sigma a/(gH)^{\frac{1}{2}},$$

and, on writing (4.3) in terms of the outer variable ξ, the same argument as before requires that $Q_a = 0$ for matching to be possible. Thus $Q_a = 0 = Q_b$ and we have finally that

$$W_0(\tilde{z}) = F. \quad (4.4)$$

The corresponding expansion for ϕ_1 can be obtained immediately from (4.2) and (4.3). Under the barrier with $\tilde{x} \ll -1$, ζ is close to unity and expanding for $\zeta-1$ small in (3.6) and (3.8) leads to

$$\phi_1 = \frac{Q'_bH}{b}\left(\tilde{x}-\frac{c}{\pi H}\ln\frac{H}{c}\right)+\frac{Q'_a}{\pi}\ln\frac{H}{b}$$
$$+F'+O(\exp(\pi H\tilde{x}/b)) \quad \text{for} \quad \tilde{x} \ll -1, \quad 0 < \tilde{y} < b/H. \quad (4.5)$$

Summarizing, we now have that near the edge (a, b)

$$\phi_0+\epsilon\phi_1 = \begin{cases} F+\epsilon\left\{\dfrac{Q'_a+Q'_b}{\pi}\left(\pi\tilde{x}+\dfrac{c}{H}\ln\dfrac{c}{b}\right)+F'+O(\exp(-\pi\tilde{x}))\right\} \quad \text{for} \quad \tilde{x} \gg 1, & (4.6a) \\ F+\epsilon\left\{\dfrac{Q'_aH}{c}\left(\tilde{x}+\dfrac{c}{\pi H}\ln\dfrac{c}{b}-\dfrac{b}{\pi H}\ln\dfrac{H}{b}\right)+\dfrac{Q'_b}{\pi}\ln\dfrac{H}{b}+F'+O(\exp(\pi H\tilde{x}/c))\right\} & \\ \qquad\qquad \text{for} \quad \tilde{x} \ll -1, \quad b/H < \tilde{y} < 1, & (4.6b) \\ F+\epsilon\left\{\dfrac{Q'_a}{\pi}\ln\dfrac{H}{b}+\dfrac{Q'_bH}{b}\left(\tilde{x}-\dfrac{c}{\pi H}\ln\dfrac{H}{c}\right)+F'+O(\exp(\pi H\tilde{x}/b))\right\} & \\ \qquad\qquad \text{for} \quad \tilde{x} \ll -1, \quad 0 < \tilde{y} < b/H. & (4.6c) \end{cases}$$

The corresponding expansions near the other edge $(-a, b)$ are obtained from (4.6a–c) by replacing Q'_a, Q'_b, F and F' by Q'_c, Q'_d, G and G' and $\tilde{x}$ by $-\tilde{x}_1$ and are given by

$$\phi_0+\epsilon\phi_1 = \begin{cases} G+\epsilon\left\{\dfrac{Q'_c+Q'_d}{\pi}\left(-\pi\tilde{x}_1+\dfrac{c}{H}\ln\dfrac{c}{b}\right)+G'+O(\exp(\pi\tilde{x}_1))\right\} & \text{for } \tilde{x}_1 \ll -1, \quad (4.7a)\\ G+\epsilon\left\{\dfrac{Q'_cH}{c}\left(-\tilde{x}_1+\dfrac{c}{\pi H}\ln\dfrac{c}{b}-\dfrac{b}{\pi H}\ln\dfrac{H}{b}\right)\right. \\ \quad \left. +\dfrac{Q'_a}{\pi}\ln\dfrac{H}{b}+G'+O(\exp(-\pi H\tilde{x}_1/c))\right\} & \text{for } \tilde{x}_1 \gg 1, \ b/H < \tilde{y} < 1, \quad (4.7b)\\ G+\epsilon\left\{\dfrac{Q'_c}{\pi}\ln\dfrac{H}{b}+\dfrac{Q'_dH}{b}\left(-\tilde{x}_1-\dfrac{c}{\pi H}\ln\dfrac{H}{c}\right)+G'+O(\exp(-\pi H\tilde{x}_1/b))\right\} \\ & \text{for } \tilde{x}_1 \gg 1, \ 0 < \tilde{y} < b/H. \quad (4.7c)\end{cases}$$

The variables and constants in (4.6) and (4.7) will indicate which edge is being referred to.

5. The matching

The matching is performed by use of the principle outlined by Van Dyke (1964, p. 89). Writing (4.6) in terms of the outer variable X and retaining only terms of zero order (ϵ^0), we have in the various regions

$$\phi_0+\epsilon\phi_1 = \begin{cases} F+(Q'_a+Q'_b)X+O(\epsilon), & (5.1a)\\ F+\dfrac{Q'_aH}{c}(\xi-D)+O(\epsilon), \quad D = \dfrac{\sigma a}{(gH)^{\frac{1}{2}}}, & (5.1b)\\ F+\dfrac{\sigma aH}{(gH)^{\frac{1}{2}}}\dfrac{Q'_b}{b}(\xi_1-1)+O(\epsilon). & (5.1c)\end{cases}$$

This is the one-term outer limit of the two-term inner expansion and we note that $\epsilon a = \sigma aH/(gH)^{\frac{1}{2}} = O(H)$ for $2a = O(\lambda)$. The two-term inner limit of the one-term outer expansion, in terms of the outer variable X, is

$$\Phi^R = T_0+i\epsilon\tilde{x}T_0+\ldots = T_0+iXT_0+O(\epsilon), \qquad (5.2)$$

$$\Phi^U = U_0\exp[ia\sigma/(gc)^{\frac{1}{2}}]+V_0\exp[-ia\sigma/(gc)^{\frac{1}{2}}]$$
$$+i(H/c)^{\frac{1}{2}}(\xi-D)\{U_0\exp[ia\sigma/(gc)^{\frac{1}{2}}]-V_0\exp[-ia\sigma/(gc)^{\frac{1}{2}}]\}+O(\epsilon), \qquad (5.3)$$

and
$$\Phi^D = P_0(\xi_1-1)+P_0+Q_0 \qquad (5.4)$$

in the various regions, and we note that (5.4) may be taken as the entire outer solution under the plate since, the plate width being taken to be $O(\lambda)$, the summation terms in (2.10) would become arbitrarily large near $x = \pm a$, so that P_n and Q_n $(n = 1, 2, \ldots)$ must be arbitrarily small. Matching (5.2), (5.3) and (5.4) with (5.1a, b, c) gives in the various regions

$$F=T_0, \quad iT_0 = Q'_a+Q'_b, \qquad (5.5a)$$

$$F = U_0\exp(ia\kappa_0)+V_0\exp(-ia\kappa_0), \quad \frac{Q'_aH}{c} = i\left(\frac{H}{c}\right)^{\frac{1}{2}}[U_0\exp(ia\kappa_0)-V_0\exp(-ia\kappa_0)], \qquad (5.5b)$$

$$F=P_0+Q_0, \quad (a\kappa H/b)Q'_b = P_0, \qquad (5.5c)$$

where κ_0 and κ denote $\sigma/(gc)^{\frac{1}{2}}$ and $\sigma/(gH)^{\frac{1}{2}}$, the wavenumbers for waves above and away from the plate respectively. For the expansions near the edge $(-a, b)$ the corresponding development gives

$$\phi + \epsilon\phi_0 = \begin{cases} G - (Q'_c + Q'_d)\,X_1 + O(\epsilon), & (5.6a) \\ G - (Q'_c H/c)\,(\xi + D) + O(\epsilon), & (5.6b) \\ G - (a\kappa H/b)\,Q'_d(\xi_1 + 1) + O(\epsilon), & (5.6c) \end{cases}$$

and

$$\Phi^L = 1 + R_0 + i(1 - R_0)\,X_1 + O(\epsilon), \tag{5.7a}$$

$$\Phi^U = U_0 \exp(-ia\kappa_0) + V_0 \exp(ia\kappa_0) + i(H/c)^{\frac{1}{2}}\,(\xi + D)\,(U_0 \exp(-ia\kappa_0) - V_0 \exp(ia\kappa_0)] + O(\epsilon), \tag{5.7b}$$

$$\Phi^D = P_0(\xi_1 + 1) - P_0 + Q_0 + O(\epsilon). \tag{5.7c}$$

Matching now gives

$$G = 1 + R_0, \quad -Q'_c - Q'_d = i(1 - R_0), \tag{5.8a}$$

$$\left.\begin{aligned} G &= U_0 \exp(-ia\kappa_0) + V_0 \exp(ia\kappa_0), \\ -Q'_c H/c &= i(H/c)^{\frac{1}{2}}\,[U_0 \exp(-ia\kappa_0) - V_0 \exp(ia\kappa_0)], \end{aligned}\right\} \tag{5.8b}$$

$$G = -P_0 + Q_0, \quad (-a\kappa H/b)\,Q'_d = P_0. \tag{5.8c}$$

Eliminating in favour of R_0, T_0, U_0 and V_0 in (5.5) and (5.8) leaves us with

$$R_0 - \exp(-ia\kappa_0)\,U_0 - \exp(ia\kappa_0)\,V_0 = -1, \tag{5.9a}$$

$$T_0 - \exp(ia\kappa_0)\,U_0 - \exp(-ia\kappa_0)\,V_0 = 0, \tag{5.9b}$$

$$R_0 + T_0 - 2i(c/H)^{\frac{1}{2}} \sin(a\kappa_0)\,U_0 - 2i(c/H)^{\frac{1}{2}} \sin(a\kappa_0)\,V_0 = 1, \tag{5.9c}$$

$$-\left(1 - \frac{2ia\kappa H}{b}\right) R_0 + T_0 + \frac{2ia\kappa}{b}(cH)^{\frac{1}{2}} \exp(-ia\kappa_0)\,U_0 - \frac{2ia\kappa}{b}(cH)^{\frac{1}{2}} \exp(ia\kappa_0)\,V_0 = 1 + \frac{i2a\kappa H}{b}. \tag{5.9d}$$

This system can be solved for the zero-order reflexion and transmission coefficients, which are identified as R_0 and T_0 respectively. We have then

$$R_0 = \chi\{\kappa l \sin\theta - 2\mu(1 - \cos\theta)\} \tag{5.10}$$

and

$$T_0 = \chi\{2i(\sin\theta + \kappa l H\mu/b)\}, \tag{5.11}$$

where
$$\chi = 1 \bigg/ \left\{2\mu(1 - \cos\theta) + \frac{\kappa l H}{b}(1 + \mu^2)\sin\theta + 2i\left(\sin\theta + \frac{\kappa l H\mu}{b}\cos\theta\right)\right\},$$

$l = 2a$, $\theta = \kappa_0 l$ and $\mu = (c/H)^{\frac{1}{2}}$. It can be shown that $|R_0|^2 + |T_0|^2 = 1$.

6. Higher approximations

The matching scheme outlined in the previous section may be extended to any order provided of course that the higher-order solutions can be obtained explicitly. For the outer solution there is no problem as is evident from the form of the system (2.5). For the inner solution we shall content ourselves with the second-order (ϵ^2) terms only. From (3.4) we seek a harmonic function $W_2(\zeta)$ satisfying

$$\operatorname{Im}\frac{dW_2}{d\tilde{z}} = \begin{cases} -\operatorname{Re} W_0 & \text{on} \quad \tilde{y} = 1 \quad \text{or} \quad A_\infty B_\infty, \\ 0 & \text{elsewhere along the } \operatorname{Re}\zeta \text{ axis} \end{cases}$$

near the edge (a, b), where the same mapping is used as for finding W_0 and W_1. W_0 being given by (4.4) leads to

$$\frac{dW_2}{d\zeta} = -\frac{d\tilde{z}}{d\zeta}\frac{F}{\pi}\ln(\zeta+d),$$

whence we have

$$W_2 = -\frac{cF}{\pi^2 H}\int^{\zeta}\frac{\ln(u+d)}{u+d}du - \frac{bF}{\pi^2 H}\int^{\zeta}\frac{\ln(u+d)}{u-1}du. \tag{6.1}$$

It can be shown that

$$W_2 = \begin{cases} \dfrac{-F}{2\pi^2}(\ln\zeta)^2 + I + O\left(\dfrac{1}{\zeta}\ln\zeta\right) \quad \text{as} \quad |\zeta| \to \infty, \\ \dfrac{-Fc}{2\pi^2 H}\{\ln(\zeta+d)\}^2 + \dfrac{Fb^2}{\pi^2 H^2}[(\zeta+d)\ln(\zeta+d) - (\zeta+d)] + I_1 \\ \qquad + O\{(\zeta+d)^2\ln(\zeta+d)\} \quad \text{as} \quad \zeta \to -d, \end{cases}$$

where I and I_1 are arbitrary constants, which are related in principle once a suitable lower limit is chosen. The real part of W_2 yields

$$\phi_2 = \begin{cases} -\dfrac{F}{2}(x^2-\tilde{y}^2) - \tilde{x}\dfrac{Fc}{\pi H}\ln\dfrac{c}{b} + I - \dfrac{Fc^2}{2\pi^2 H^2}\left(\ln\dfrac{c}{b}\right)^2 + O(\tilde{x}\exp(-\pi x)) \quad \text{for} \quad \tilde{x} \gg 1, & (6.2a) \\ -\dfrac{FH}{2c}\left[\tilde{x}^2 - \left(\tilde{y}-\dfrac{b}{H}\right)^2\right] - \dfrac{FH}{c}\tilde{x}\left(\dfrac{c}{\pi H}\ln\dfrac{c}{b} - \dfrac{b}{\pi H}\ln\dfrac{H}{b}\right) \\ \quad + I_1 - \dfrac{FH}{2c}\left(\dfrac{c}{\pi H}\ln\dfrac{c}{b} - \dfrac{b}{\pi H}\ln\dfrac{H}{b}\right)^2 \\ \quad + O(x\exp(\pi H\tilde{x}/c)) \quad \text{for} \quad \tilde{x} \ll -1, \quad b/H < \tilde{y} < 1. & (6.2b) \end{cases}$$

Near the edge $(-a, b)$, ϕ_2 is given by (6.2) with F, I and I_1 replaced by G, J and J_1, say, and $\tilde{x}$ replaced by $-\tilde{x}_1$. It can be shown that the matching up to terms involving ϵ^2 follows through easily, but it is evident that the algebra will consistently become more difficult as one goes to higher orders. We shall not pursue this further.

7. Discussion

We note that Φ^U as given by (2.8) is valid provided that $\mu \equiv (c/H)^{\frac{1}{2}} \neq 0$. If $\mu = 0$, the plate is on the surface and from (5.10), on letting $\mu \to 0$ with κl fixed, we obtain

$$|R_0| = \kappa a/(\kappa^2 a^2+1)^{\frac{1}{2}} \quad \text{for} \quad \mu = 0, \quad \theta \neq n\pi, \quad n = 0, 1, 2, \ldots . \tag{7.1}$$

This is the first-order reflexion coefficient obtained by Wells (1953) for the rigid floating dock. Returning to (5.10) we find that, for μ sufficiently small that $\mu^2 \ll 1$ and $\theta = 2n\pi + \epsilon$, $n = 1, 2, \ldots$, $|\epsilon| \ll 1$,

$$|R_0| = \frac{n\pi\mu|\epsilon|}{\{n^2\pi^2\mu^2\epsilon^2 + [\epsilon + 2n\pi\mu^2]^2\}^{\frac{1}{2}}} + O(\epsilon^2),$$

whereas when $\theta = (2n+1)\pi + \epsilon$

$$|R_0| = \frac{|(2n+1)\pi\mu\epsilon + 4\mu|}{\{[4\mu - (2n+1)\pi\mu\epsilon]^2 + 4\{\epsilon + (2n+1)\pi\mu^2]^2\}^{\frac{1}{2}}} + O(\epsilon^2).$$

These features are illustrated in figure 2, which gives a plot of $|R_0|$ for a surface plate ($\mu = 0$) and a plate very close to the surface ($\mu = 0{\cdot}005$).

Another interesting aspect of the solution is the very simple form of the expressions for the reflexion and transmission coefficients given by (5.10) and (5.11), which makes

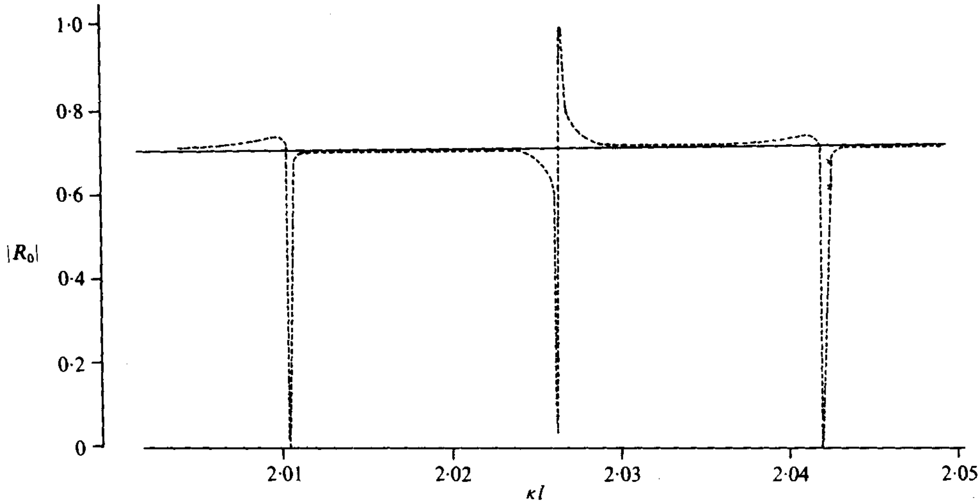

FIGURE 2. Plot of $|R_0|$ against κl. ——, $\mu = 0$; - - -, $\mu = 0{\cdot}005$. (For $\mu = 0{\cdot}005$, $\theta = \kappa l/\mu = 128\pi$ at $\kappa l = 2{\cdot}01062$ and $\theta = 130\pi$ at $\kappa l = 2{\cdot}04204$.)

it easy to check the energy balance. As another check, we may consider the sill mound, which is equivalent to assuming zero flux below the plate. On putting $P_0 = 0$ in (5.4), (5.5), (5.7) and (5.8) and solving the resultant equations for R_0 we find

$$|R_0|^2 = \frac{(1-\mu^2)^2}{4\mu^2 \cot^2\theta + (1+\mu^2)^2}, \tag{7.2}$$

where θ and μ are as defined after (5.11). Equation (7.2) is the expression first obtained by Rayleigh (1945, vol. 2, p. 87) in his discussion of the analogous problem of the reflexion of waves from a plate of finite thickness.

Further, in the limit of a semi-infinite barrier, the leading-order terms R_0 and U_0 may be obtained from (5.8). Assuming the barrier to occupy the position $0 < x < \infty$, $y = b$, we have from (2.10) that P_n $(n = 0, 1, \ldots)$ must be zero for the solution to be bounded, and (5.8) then gives

$$1 + R_0 = U_0 = i(H/c)^{\frac{1}{2}} Q_c' = (H/c)^{\frac{1}{2}} (1 - R_0),$$

whence

$$R_0 = \frac{\kappa_0 - \kappa}{\kappa_0 + \kappa}, \quad T_0 = \frac{2\kappa_0}{\kappa_0 + \kappa}. \tag{7.3}$$

These are the limiting forms of the reflexion and transmission coefficients for $\lambda \gg H$ and may be derived from the results of Heins (1950).

8. Oblique incidence

The analysis in §§2–5 may be easily adapted to a special case of oblique incidence. Under the assumptions that the wavelength λ is large compared with the depth of the channel and that the variation of Φ in the lateral (z) direction is harmonic, the linear model reduces to

$$\left.\begin{aligned} \Phi_{xx} + \Phi_{yy} - k^2\Phi = 0, \\ \Phi_y - (\sigma^2/g)\,\Phi = 0 \quad \text{on} \quad y = H, \\ \Phi_y = 0 \quad \text{on the plate and channel floor,} \end{aligned}\right\} \tag{8.1}$$

where we have replaced $\Phi(x,y,z)$ by $\Phi(x,y)\exp(ikz)$, the plate occupies

$$|x| < a, \quad y = b, \quad -\infty < z < \infty,$$

the Oz axis is parallel to the edge of the plate, and the projection in the Oxy plane is as described in §2. For $\lambda \gg H$ we again define a small parameter $\epsilon = \sigma(H/g)^{\frac{1}{2}}$, whence $kH < \rho_0 H \ll 1$, ρ_0 being the positive real root of the equation

$$\rho \sinh \rho H - (\sigma^2/g) \cosh \rho H = 0.$$

If we assume the outer expansion to be given by (2.3) the solutions for Φ_0 and Φ_1 are of the same form as before for the regions $x \gg a$, $|x| < a$ (above the plate) and $x \ll -a$. For the region under the plate it is more convenient to let $\xi_1 = x/a$ and $\eta_1 = y/b$. Then, confining ourselves again to the case $a = O(\lambda)$, we find that $b/a = O(\epsilon)$ and hence Φ_0^D takes the form $P_0'\xi_1 + Q_0'$, where P_0' and Q_0' are arbitrary constants. The inner solution again takes the same form as in §3, and the leading values for the reflexion and transmission coefficients are given by (5.10) and (5.11) with κ replaced by $\sigma/(gH)^{\frac{1}{2}}$ and κ_0 by $\sigma/(gc)^{\frac{1}{2}}$.

The scheme does not, however, allow us to calculate higher approximations since one would need to define the ratio k/ρ_0 more precisely before Φ_2 could be determined. For the inner region the powerful method of conformal mapping cannot be used since ϕ_2 would not then satisfy Laplace's equation.

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